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# Multiplicity and invariants in birational geometry

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# Multiplicity and invariants in birational geometry

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## Definition 1

Let  $X$  be a variety and let  $x$  be a point of  $X$ . The multiplicity of  $x$  on  $X$ , denoted  $\text{mult}_x X$ , means  $\text{mult}_{O_{X,x}}$ .

For a normal variety  $X$ , a prime divisor  $E$  over  $X$  means that a prime divisor  $E$  appears on a resolution  $f: Y \rightarrow X$ . Let  $E$  be a prime divisor over  $X$  and appear on a log resolution  $f: Y \rightarrow X$  of  $X$ . The log discrepancy of  $X$  with respect to  $E$  is

$$a_E(X) := \text{ord}_E(K_Y - f^*K_X) + 1.$$

For a closed subset  $Z$  of  $X$ , the minimal log discrepancy  $\text{mld}_Z(X)$  over  $Z$  is the infimum of  $a_E(X)$  for all prime divisor  $E$  over  $X$  with center in  $Z$ .

Let  $I \subset O_X$  be a nonzero ideal. The log-canonical threshold of  $I$  in  $X$  is defined as follows:

$$\text{lct}(I) = \sup\{c \in \mathbb{Q} \mid (X, I^c) \text{ is log canonical}\}.$$

## Conjecture 1

Let  $X$  be an  $n$ -dimensional locally a complete intersection variety. Then  $\text{mult}_x X \leq 2^{n-\text{mld}_x X}$  for a closed point  $x$  of  $X$  and equality holds if and only if  $\text{emb}(X, x) = 2n - \text{mld}_x X$ .

## Conjecture 2

Let  $X$  be an  $n$ -dimensional locally a complete intersection variety with log canonical singularities. Then  $\text{mult}_x X \leq 2^{n-\lceil \text{lct}(m_x) \rceil}$  for a closed point  $x$  of  $X$  and equality holds if and only if  $\text{emb}(X, x) = 2n - \lceil \text{lct}(m_x) \rceil$ .

## Watanabe's Conjecture

Let  $X$  be an  $n$ -dimensional locally a complete intersection variety with canonical singularities. Then  $\text{mult}_x X \leq 2^{n-1}$  for a closed point  $x$  of  $X$  and equality holds if and only if  $\text{emb}(X, x) = 2n - 1$ .

## Remark

- $\lceil \text{lct}(m_x) \rceil \leq \text{mld}_x X$
- Conjecture 1  $\Rightarrow$  Conjecture 2  $\Rightarrow$  Watanabe's Conjecture
- Watanabe's Conjecture is a generalization of Proposition 3.1 in [W].

## Proposition 1

Let  $X$  be an  $n$ -dimensional Gorenstein variety with canonical singularities and let  $x$  in  $X$  be a closed point. Let  $\text{emb}(X, x) = v$  and  $\lceil \text{lct}(m_x) \rceil = c$ .

(1) If  $n + 1 - c = 2r + 1$ , then

$$\text{mult}_x X \leq \binom{v-n+r}{r} + \binom{v-n+r-1}{r-1}.$$

(2) If  $n + 1 - c = 2r$ , then

$$\text{mult}_x X \leq 2 \binom{v-n+r-1}{r-1}.$$

## Remark

This Proposition 1 is a generalization of Theorem 3.1 in [HW] of characteristic zero. But our proof is just an imitation of the proof of Theorem 3.1 in [HW].

## Proposition 2

Let  $X$  be an  $n$ -dimensional normal locally complete intersection variety with canonical singularities.

If  $n \leq 5$ ,  $\text{mult}_x X \leq 2^{n-\text{mld}_x X}$  for a closed point  $x$  of  $X$ .

If  $n \leq 4$ ,  $\text{mult}_x X \leq 2^{n-\text{mld}_x X}$  for a closed point  $x$  of  $X$  and equality holds if and only if  $\text{emb}(X, x) = 2n - \text{mld}_x X$ .

## Theorem 1

Let  $X$  be an  $n$ -dimensional locally a complete intersection variety with canonical singularities. Let  $\lceil \text{lct}(m_x) \rceil = c$ . Let  $e = \text{emb}(X, x) - n$ . Let  $c + 2e - n > 0$ .

(1) If  $n - c + 1 = 2r$ , then

$$\text{mult}_x X \leq 2 \sum_{j=0}^{c+2e-n} \binom{c+2e-n}{j} \binom{n-c-e+r-j-1}{r-j-1}.$$

(2) If  $n - c + 1 = 2r + 1$ , then

$$\text{mult}_x X \leq \sum_{j=0}^{c+2e-n} \binom{c+2e-n}{j} \left\{ \binom{n-c-e+r-j-1}{r-j-1} + \binom{n-c-e+r-j}{r-j} \right\}.$$

## Key idea

By the formula (13) in [IR], we obtain

$$\text{mld}_x X = \inf_m \{ (m+1)n - \dim \pi_m^{-1}(x) \}, \text{ where}$$

$\pi_m: X_m \rightarrow X$  is a truncation map. By this equality, we get a regular sequence consisting the initial terms of some defining polynomials of  $X$ . Using this fact and Proposition 1, I proved Proposition 2 and Theorem 1.

## Corollary 1

Let  $X$  be an  $n$ -dimensional locally a complete intersection variety with canonical singularities. If  $n \leq 32$ , then

$\text{mult}_x X \leq 2^{n-\lceil \text{lct}(m_x) \rceil}$  for a closed point  $x$  of  $X$  and equality holds if and only if  $\text{emb}(X, x) = 2n - \lceil \text{lct}(m_x) \rceil$ .

## Corollary 2

Watanabe's Conjecture is true if  $n \leq 32$ .

## Theorem 2

Let  $X$  be an  $n$ -dimensional locally a complete intersection simplicial toric variety. Then  $\text{mld}_x X = \lceil \text{lct}(m_x) \rceil$  for a fixed closed point  $x$  of  $X$ .

## Theorem 3

Let  $X$  be an  $n$ -dimensional locally a complete intersection simplicial toric variety. Then  $\text{mult}_x X \leq 2^{n-\text{mld}_x X}$  for a fixed closed point  $x$  of  $X$  and equality holds if and only if  $\text{emb}(X, x) = 2n - \text{mld}_x X$ .

## Key idea

Nakajima classified cones corresponding to locally a complete intersection toric varieties in [N]. Using this classification, I proved Theorem 2 and Theorem 3.

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